HODGE STRUCTURES AND WEIERSTRASS σ-FUNCTION

GRZEGORZ BANASZAK* AND JAN MILEWSKI**

Abstract. In this paper we introduce new definition of Hodge structures and show that $\mathbb{R}$-Hodge structures are determined by $\mathbb{R}$-linear operators that are annihilated by the Weierstrass σ-function

1. Introduction

Classically a Hodge structure of a given weight can be defined in the four equivalent ways as follows (see e.g. [2], [5]):

Definition 1.1. A Hodge structure of a weight n on a real vector space V consists of a finite–dimensional $\mathbb{R}$–vector space V together with any of the following equivalent data:

(i) A decomposition $V_{\mathbb{C}} = \oplus_{p+q=n} V^{p,q}$, called the *Hodge decomposition*, such that $\overline{V^{p,q}} = V_{q,p}$.

(ii) A decreasing filtration $F_H^r V_{\mathbb{C}}$ of $V_{\mathbb{C}}$, called the *Hodge filtration*, such that $F_H^r V_{\mathbb{C}} \oplus \overline{V_{\mathbb{C}}^{n-r+1}} = V_{\mathbb{C}}$.

(iii) A homomorphism $h_n : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ of real algebraic groups, and also specifying that the weight of the Hodge structure is n.

(iv) A homomorphism $h_n : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ of real algebraic groups such that via the decomposition $\mathbb{G}_m/\mathbb{R} \to \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ an element $t \in \mathbb{G}_m/\mathbb{R}$ acts as $t^{-n} \cdot Id$.

Throughout the paper we work with Hodge structures of various weights, hence by a Hodge structure we understand here a finite direct sum

$$\rho \ := \ \bigoplus_{j=1}^{k} h_{n_j} \tag{1}$$

of representations h_{n_j} described in (iii) or (iv) of the Definition 1.1.

In this paper we consider Hodge structures on real vector space V via representations of the Lie algebra of the real algebraic group $\mathbb{S}$ (denoted also $\mathbb{C}^{\times}$) on V. In section 2 we show that a Hodge structure can be treated as a pair of operators E, T na V satisfying certain conditions (see Theorem 2.1). In section 3 we show that a Hodge structure can be treated as a single operator $S := E + T$ on V such that $\sigma(S) = 0$ for a Weierstrass σ-function which corresponds to decomposition of V into eigenspaces of operators E and T. Weirstrass σ-function does not have multiple zeros hence this corresponds to the fact that complexification of S does not have generalized eigenvectors other than ordinary ones.

2010 *Mathematics Subject Classification.* 14D07.

Key words and phrases. Hodge structure, Weierstrass σ function.

*Partially supported by the NCN (National center of Science for Poland) NN201 607440

**Partially supported by the NCN grant NN201 373236.

2. Hodge structures and Lie algebras.

The following theorem gives another definition of the Hodge structure.

Theorem 2.1. *Let V be a finite dimensional vector space over $\mathbb{R}$. There is a one to one correspondence between the family of Hogde structures on V and the family of pairs of endomorphisms $E, T \in \mathrm{End}_{\mathbb{R}}(V)$ satisfying the following conditions:*

$$[E,T]=0, \quad \sin(\pi E)=0, \quad \sinh(\pi T)=0, \tag{2}$$

$$\sin(\frac{\pi}{2}(E^2+T^2))=0 \tag{3}$$

Proof. Consider a Hodge structure on V. By (1) (cf. Definition 1.1 (iii)) this gives a representation:

$$\rho : \mathbb{S} \to \mathrm{GL}(V)$$

of real algebraic groups. The representation ρ decomposes into irreducible representations $\rho_{p,q}$ with multiplicities $m_{p,q}$

$$\rho = \bigoplus_{q \leq p} m_{p,q}\ \rho_{p,q},$$

$$\rho_{p,q}(re^{i\phi}) \ := \ r^{p+q} \begin{bmatrix} \cos(p-q)\phi & -\sin(p-q)\phi \\ \sin(p-q)\phi & \cos(p-q)\phi \end{bmatrix} \quad \text{for } \ p \neq q, \ p,q \in \mathbb{Z}$$

$$\rho_{p,p}(re^{i\phi}) \ := \ r^{2p} \begin{bmatrix} 1 \end{bmatrix}.$$

Certainly, the complexification of the representation $\rho_{p,q}$ for $q < p$ decomposes into two one-dimensional $\mathbb{C}$-vector spaces:

$$\rho_{p,q} \otimes_{\mathbb{R}} \mathbb{C} = \rho_{p,q}^{\mathbb{C}} \oplus \rho_{q,p}^{\mathbb{C}}, \tag{4}$$

where

$$\rho_{m,n}^{\mathbb{C}}(z) = z^m \overline{z}^n. \tag{5}$$

Consider the real Lie algebra representation (the derivative of ρ):

$$\mathcal{L}(\rho) : \mathbb{C} \to \mathrm{End}(V).$$

For $q \leq p$ the representation $\mathcal{L}(\rho_{p,q})$ is also two-dimensional

$$\mathcal{L}(\rho_{p,q})(1) = (p+q)I \ \text{ and } \ \mathcal{L}(\rho_{p,q})(i) = (p-q)J,$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

For $p = q$

$$\mathcal{L}(\rho_{p,p})(1) = 2p \ \text{ and } \ \mathcal{L}(\rho_{p,p})(i) = 0.$$

If we put $E := \mathcal{L}(\rho)(1)$ and $T := \mathcal{L}(\rho)(i)$ then we get equations (2) and (3). The condition (3) is fulfiled because $p-q$ and $p+q$ have the same parity.

Now let us assume that conditions (2) and (3) hold. Observe that $\sinh(z)$ and $\sin(z)$ have single zeros in the complex plane. Moreover (2) and (3) imply that the complexifications $E \otimes 1$ and $T \otimes 1 \in \mathrm{End}(V \otimes_{\mathbb{R}} \mathbb{C})$ have common eigenbasis. From this it follows that the endomorphisms $E, T \in \mathrm{End}_{\mathbb{R}}(V)$ have common Jordan decomposition into eigenspaces of dimension 1 or 2. We define a representation

$$\rho : \mathbb{C}^{\times} \to \mathrm{GL(V)},$$

$$\rho(e^{x+iy}) = \exp(xE + yT) \quad \text{for} \quad x, y \in \mathbb{R}.$$

ρ is an algebraic representation, because the equality (3) holds. The representation ρ gives the Hodge structure on V. □

3. Hodge structures via single operator

Let $\sigma(z)$ be the Weierstrass' sigma function for the lattice generated by $\omega_1 = 1-i$ and $\omega_2 = 1+i$:

$$\sigma(z) := z \prod_{(k_1,k_2)\neq(0,0)} \left(1 - \frac{z}{k_1\omega_1 + k_2\omega_2}\right) \exp\left[\frac{z}{k_1\omega_1 + k_2\omega_2} + \frac{1}{2}\left(\frac{z}{k_1\omega_1 + k_2\omega_2}\right)^2\right]$$

Theorem 3.1. *For operators $E, T \in \mathrm{End}_{\mathbb{R}}(V)$ considered above let $S := E + T$. We get the following equality*

$$\sigma(S) = 0. \tag{6}$$

Conversely every $S \in \mathrm{End}_{\mathbb{R}}(V)$ satisfying condition (6) gives a unique pair (E, T) of operators in $\mathrm{End}_{\mathbb{R}}(V)$ such that $S = E + T$ and the conditions (2) and (3) hold.

Proof. It is clear that $S = E+T$ satisfies the equation (6). Conversely, assume that an operator $S \in End_{\mathbb{R}}(V)$ satisfies (6). Since the σ function has zeros of order 1, we observe that the complexification of S is diagonalizable. We get the operators E and T considering equation

$$S(v) = \lambda v \tag{7}$$

in the complexification of V. The eigenvalues have integer real and imaginary parts with the same parity:

$$\lambda = a + ib, \qquad a, b \in \mathbf{Z}, \qquad a - b \in 2\mathbf{Z}. \tag{8}$$

Moreover we define the operators E, T in such a way that their complexifications acting on the eigenvector v of S have form: $E(v) = av$ i $T(v) = ibv$ where $S(v) = (a + ib)v$. Operators E and T satisfy equations (2) and (3). The operators E and T are uniquelly determined. Indeed, if $S = E' + T'$ such that E' and T' satisfy (2) and (3) then it is clear that $[E', S] = 0$ and $[T', S] = 0$. □

Remark 3.2. For certain Hodge structures the set of eigenvalues of the comlexification of S has further obstructions beyond (8). In this case S satisfies the equation $g(S) = 0$, where $g(z)$ is an analytic function that divides $\sigma(z)$ in such a way that $\frac{\sigma(z)}{g(z)}$ is also an analytic function on the whole complex plane.

Remark 3.3. In our work in progress we define certain deformations of Hodge structures that arise in a natural way in mathematical physics (see [1], [3], [4]).

References

[1] G. Banaszak, J. Milewski, Hodge structures in toplogical quantum mechanics, *J. Phys. Conf. Ser.* **213**, 012017,(2010).

[2] B. Gordon, A survey of the Hodge conjecture for abelian varieties, Appendix B in "A survey of the Hodge conjecture", by J. Lewis, 297–356 (1999) *American Mathematical Society*

[3] J. Milewski, Holomorphons and the standard almost complex structure on S^6, *Commentationes Mathematicae*, **XLVI (2)** (2006), 245–254.

[4] J. Milewski, Holomorphons on spheres, *Commentationes Mathematicae*, **B XLVIII (2)**),13-22, (2008).

[5] C. Peters, J. Steenbrink, Mixed Hodge structures *Ergebnisse der Math. Springer* **52**, (2008).

Department of Mathematics and Computer Science, Adam Mickiewicz University, Poznań 61614, Poland

E-mail address: banaszak@amu.edu.pl

Institute of Mathematics, Poznań University of Technology, ul. Piotrowo 3A, 60-965 Poznań, Poland

E-mail address: jsmilew@wp.pl